Gaussian Radial-Basis-Function solution of the non-linear Fokker-Planck equation

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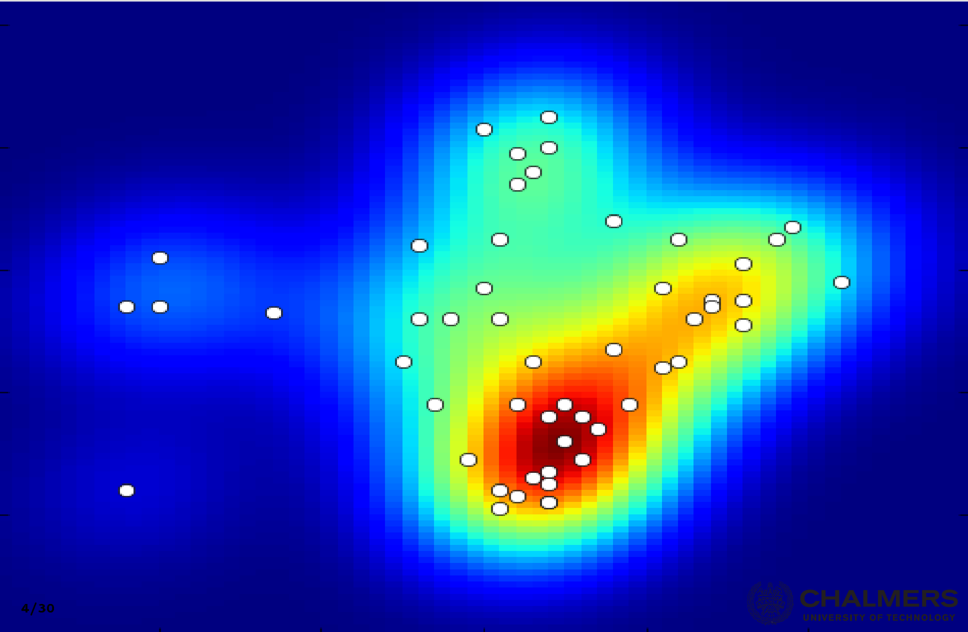
UNIVERSITY OF TECHNOLOGY
GENERAL ATOMICS

Outline

1. What are Radial Basis Functions (RBFs)?
2. Discretization of the Collision operator
3. Non-linear relaxation problem
4. What about the Vlasov-Maxwell part?
5. Summary and where to proceed

RBFs common in interpolation of scattered data



Recipe

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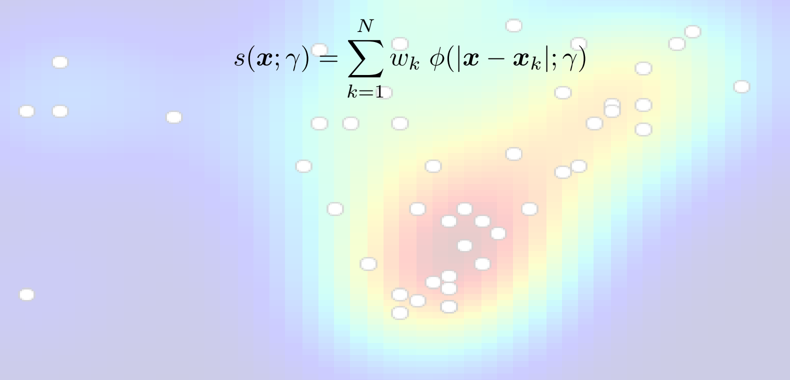
3. **Define an RBF interpolant** $f(\boldsymbol{x}) \approx s(\boldsymbol{x};\gamma)$

$$s(\boldsymbol{x};\gamma) = \sum_{k=1}^{N} w_k \, \phi(|\boldsymbol{x} - \boldsymbol{x}_k|;\gamma)$$



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$$A_{ij} w_j = f_i \quad \textbf{where} \quad A_{ij} = \phi(|\boldsymbol{x}_i - \boldsymbol{x}_j|; \gamma)$$

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5. Evaluate the interpolant $s(\boldsymbol{x}; \gamma)$ where you wish

Many different choices for basis functions

Gaussian: $\phi(|\boldsymbol{x}|;\gamma) = e^{-(\gamma|\boldsymbol{x}|)^2}$

Polyharmonic Spline: $\phi(|\boldsymbol{x}|;\gamma) \;= \gamma|\boldsymbol{x}|,\, (\gamma|\boldsymbol{x}|)^3,\, \ldots$

Multiquadric: $\phi(|\boldsymbol{x}|;\gamma) = \sqrt{1+(\gamma|\boldsymbol{x}|)^2}$

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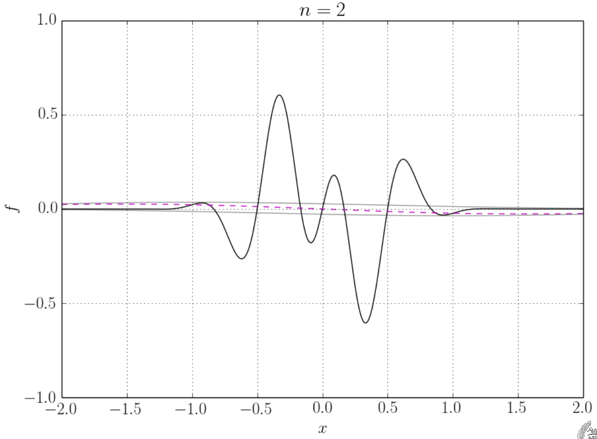

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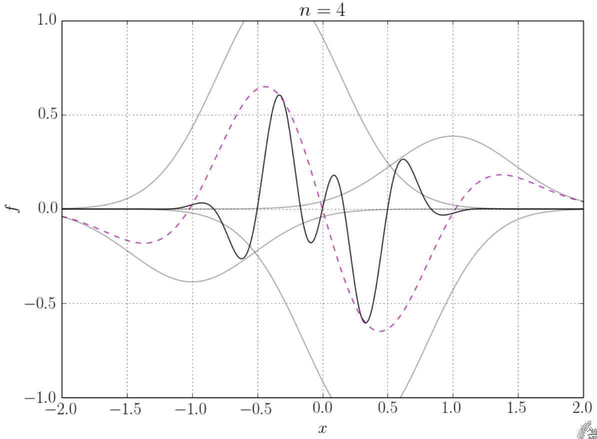

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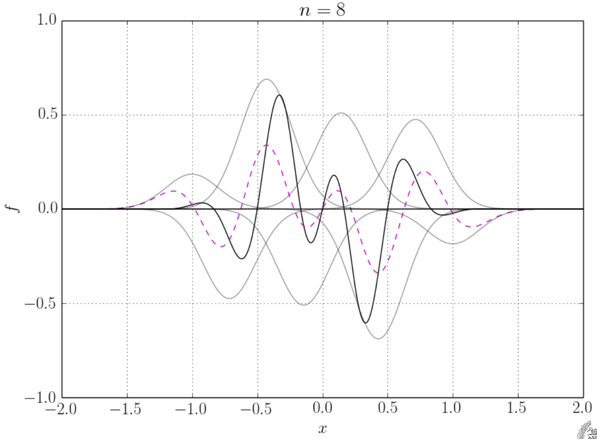

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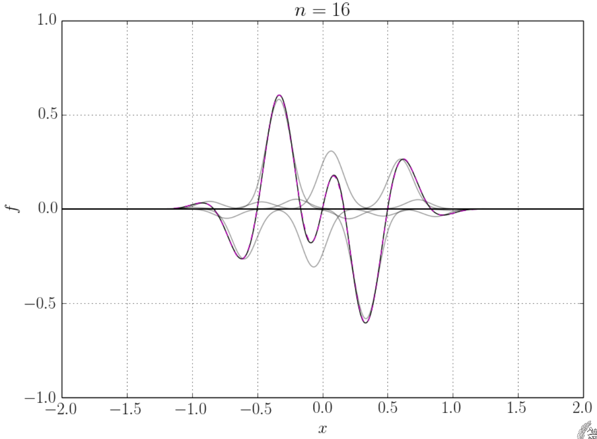

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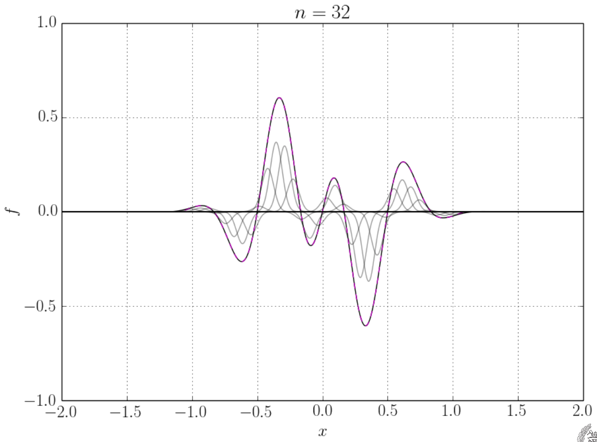

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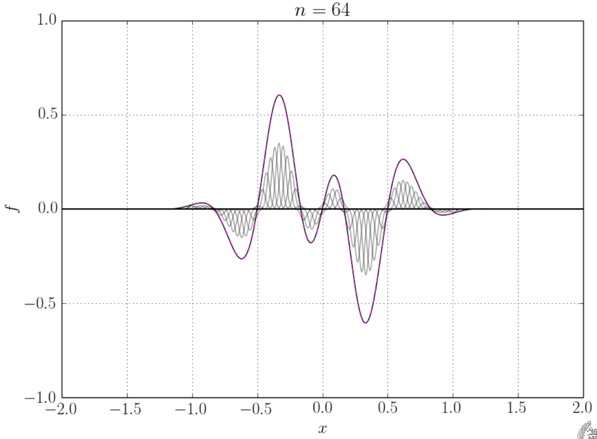

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Statistical description of collisions in plasmas

1. **Fokker-Planck operator:**

$$C_{s\bar{s}}[f_s, f_{\bar{s}}] = \frac{\partial}{\partial v^i}\left(D^{ij}_{s\bar{s}} \frac{\partial f_s}{\partial v^j} - K^i_{s\bar{s}} f_s \right)$$

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2. **Friction and diffusion coefficients**

$$K^i_{s\bar{s}} = -\gamma_{s\bar{s}}\frac{m_s}{m_{\bar{s}}}\frac{\partial \phi_{\bar{s}}}{\partial v^i}, \quad D^{ij}_{s\bar{s}} = -\gamma_{s\bar{s}}\frac{\partial^2 \psi_{\bar{s}}}{\partial v^i \partial v^j}, \quad \gamma_{s\bar{s}} = \left(\frac{e_s e_{\bar{s}}}{m_s \epsilon_0}\right)^2 \ln\Lambda.$$

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3. **Rosenbluth potentials**

$$\phi_{\bar{s}}(\boldsymbol{v}) = -\frac{1}{4\pi}\int d\bar{\boldsymbol{v}}\, f_{\bar{s}}(\bar{\boldsymbol{v}})|\boldsymbol{v} - \bar{\boldsymbol{v}}|^{-1},$$

$$\psi_{\bar{s}}(\boldsymbol{v}) = -\frac{1}{8\pi}\int d\bar{\boldsymbol{v}}\, f_{\bar{s}}(\bar{\boldsymbol{v}})|\boldsymbol{v} - \bar{\boldsymbol{v}}|,$$

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3. **Rosenbluth potentials are difficult to compute**

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A special case relaxes difficulties

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$$f(\boldsymbol{v}; \gamma, \boldsymbol{v}_d) = (\gamma/\pi)^{3/2} \exp[-\gamma(\boldsymbol{v} - \boldsymbol{v}_d)^2]$$

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$$\varphi(\boldsymbol{v};\gamma,\boldsymbol{v}_d) = -\frac{\sqrt{\gamma}}{4\pi}\Phi(\sqrt{\gamma}|\mathbf{v}-\mathbf{v}_d|),$$
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$$\Phi(y) = \frac{\operatorname{erf}(y)}{y},$$

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Our idea for easier computations

1. **Express the distribution function as a** *convolution*

$$f_s(\boldsymbol{v},t) = \int \left(\frac{\gamma}{\pi}\right)^{3/2} \exp[-\gamma\,(\boldsymbol{v}-\boldsymbol{u})^2] W_s(\boldsymbol{u},\gamma,t)\, d\boldsymbol{u}\, d\gamma,$$

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3. **Define the weight function**

$$\text{(3-D)}\; W_s(\boldsymbol{u},\gamma,t) = \sum_i w_s^i(t)\,\delta(\boldsymbol{u}-\boldsymbol{v}_s^i)\,\delta(\gamma-\gamma_s^i),$$

$$\text{(2-D)}\; W_s(\boldsymbol{u},\gamma,t) = \sum_i w_s^i(t)\frac{1}{2\pi v_{s,\perp}^i}\delta(u_\parallel - v_{s,\parallel}^i)\delta(u_\perp - v_{s,\perp}^i)\delta(\gamma-\gamma_s^i)$$

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We get a “mesh-free” approach

1. **Choice of weight function gives the RBF basis functions**

$$\text{(3-D)}\ F_s^i(\boldsymbol{v}) = \left(\gamma_s^i/\pi\right)^{3/2} \exp\left[-\gamma_s^i(\mathbf{v}-\mathbf{v}_s^i)^2\right]$$

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What about time evolution?

Collisional evolution

1. **From the continuous representation**

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3. **with an analytical coefficient**

$$C_{s\bar{s}}^{k\ell}(\boldsymbol{v}) = \gamma_{s\bar{s}} \left[\frac{m_s}{m_{\bar{s}}} F_s^k F_{\bar{s}}^\ell + \mu_{s\bar{s}} \frac{\partial \varphi_{\bar{s}}^\ell}{\partial \boldsymbol{v}} \cdot \frac{\partial F_s^k}{\partial \boldsymbol{v}} - \frac{\partial^2 \psi_{\bar{s}}^\ell}{\partial \boldsymbol{v} \partial \boldsymbol{v}} : \frac{\partial^2 F_s^k}{\partial \boldsymbol{v} \partial \boldsymbol{v}} \right],$$

where $\mu_{s\bar{s}} = m_s/m_{\bar{s}} - 1$

Equation for the weights

1. Apply center collocation, i.e., evaluate the expanded equation at the points v_s^i

$$\sum_j \mathcal{M}_s^{ij} \frac{\partial w_s^j}{\partial t} = \sum_{k,\ell,\bar{s}} w_s^k \, w_{\bar{s}}^\ell \, C_{s\bar{s}}^{ik\ell},$$

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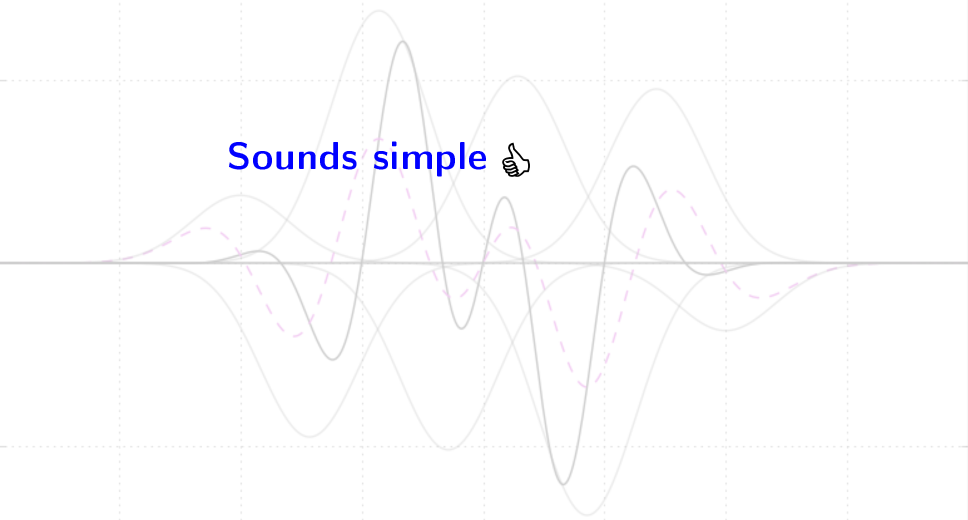
We chose center collocation because of its simplicity. Galerkin projection could also be worth considering.

The RBF method



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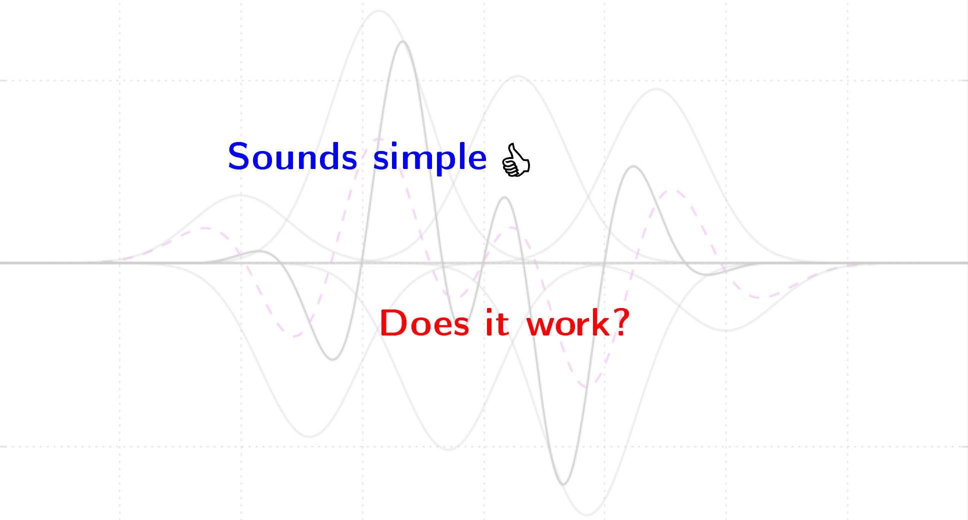

Sounds simple 👍

Does it work?

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Non-linear relaxation problem

1. **Single plasma species** s

$$C_{ss}[f_s, f_s] = \gamma_{ss} \left[f_s f_s - \frac{\partial^2 \psi_s}{\partial \boldsymbol{v} \partial \boldsymbol{v}} : \frac{\partial^2 f_s}{\partial \boldsymbol{v} \partial \boldsymbol{v}} \right],$$

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$$\frac{\partial f_s}{\partial \tau} = f_s f_s - \frac{\partial^2 \psi_s}{\partial \boldsymbol{v}\partial \boldsymbol{v}} : \frac{\partial^2 f_s}{\partial \boldsymbol{v}\partial \boldsymbol{v}},$$

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$$\tau = \gamma_{ss} t = \left(\frac{e_s^2}{m_s \epsilon_0}\right)^2 \ln \Lambda \, t$$

3. **Solve the initial value problem by applying the RBF expansion**

$$\frac{\partial f_s}{\partial \tau} = f_s f_s - \frac{\partial^2 \psi_s}{\partial \boldsymbol{v} \partial \boldsymbol{v}} : \frac{\partial^2 f_s}{\partial \boldsymbol{v} \partial \boldsymbol{v}},$$

Initial state

1. **Choose a non-linear double peaked distribution function**

$$f_s(\boldsymbol{v}, \tau_0) = \sum_{i=1}^{2} \exp[-\beta_i(\boldsymbol{v} - \boldsymbol{v}^i)^2]$$

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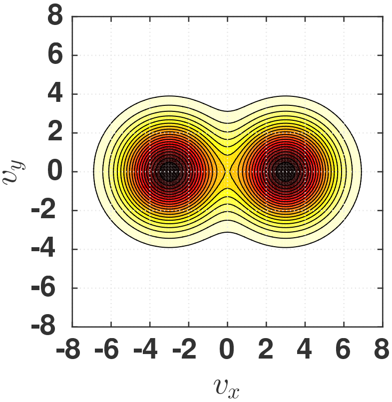
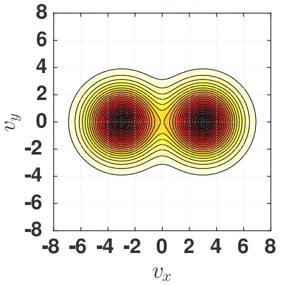


Time slices $\tau = 0, 3, 6, 40$



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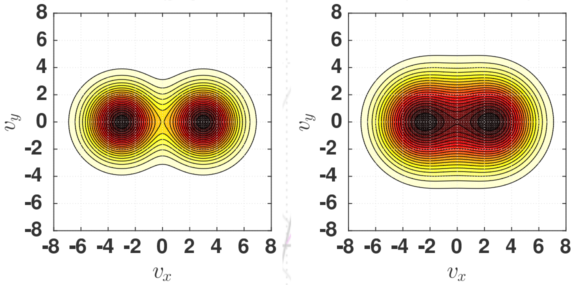

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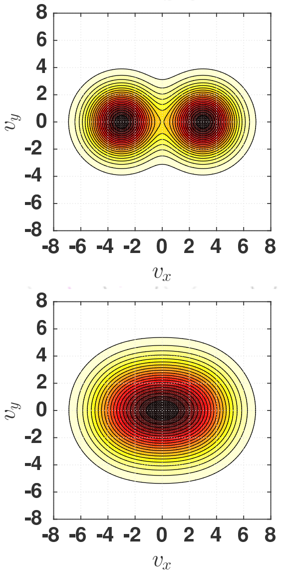

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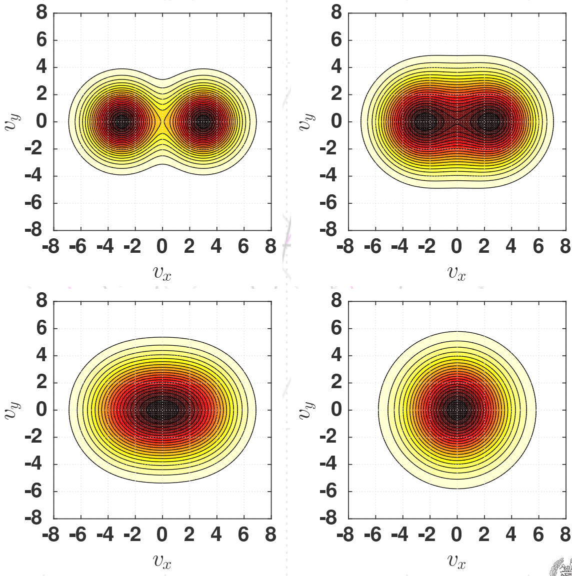
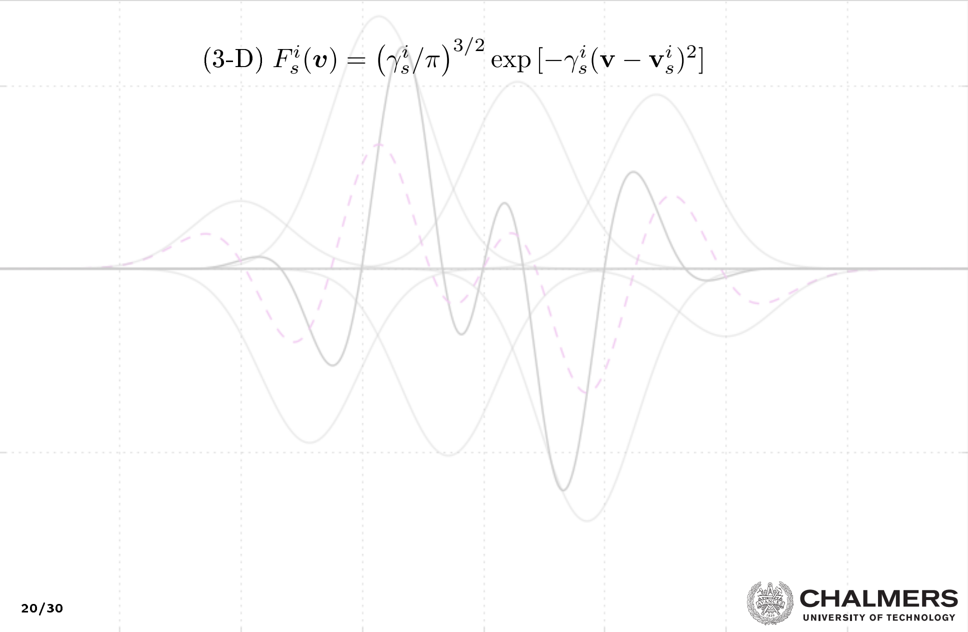


3-D vs 2-D RBF

$$(\text{3-D})\ F_s^i(\boldsymbol{v}) = \left(\gamma_s^i/\pi\right)^{3/2} \exp\left[-\gamma_s^i(\mathbf{v} - \mathbf{v}_s^i)^2\right]$$



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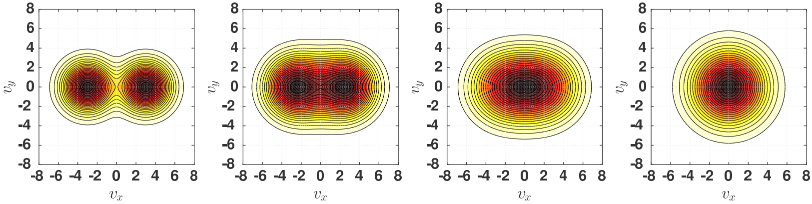

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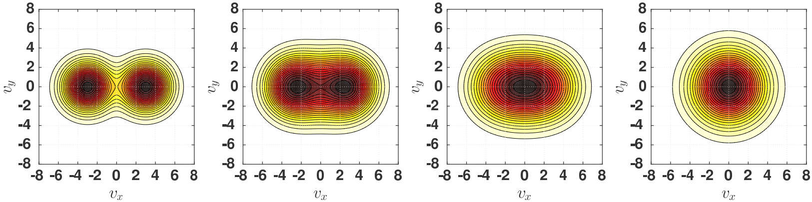

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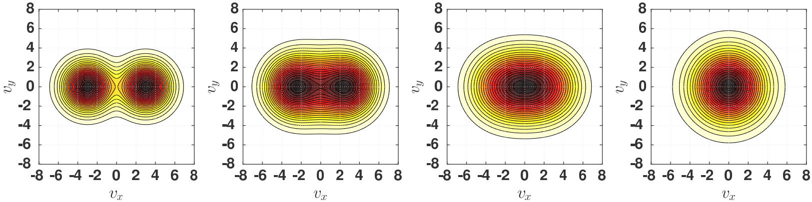

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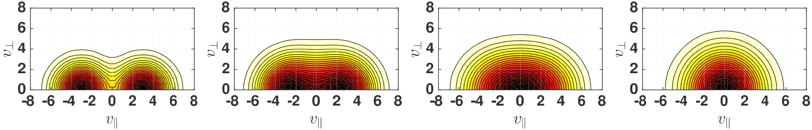
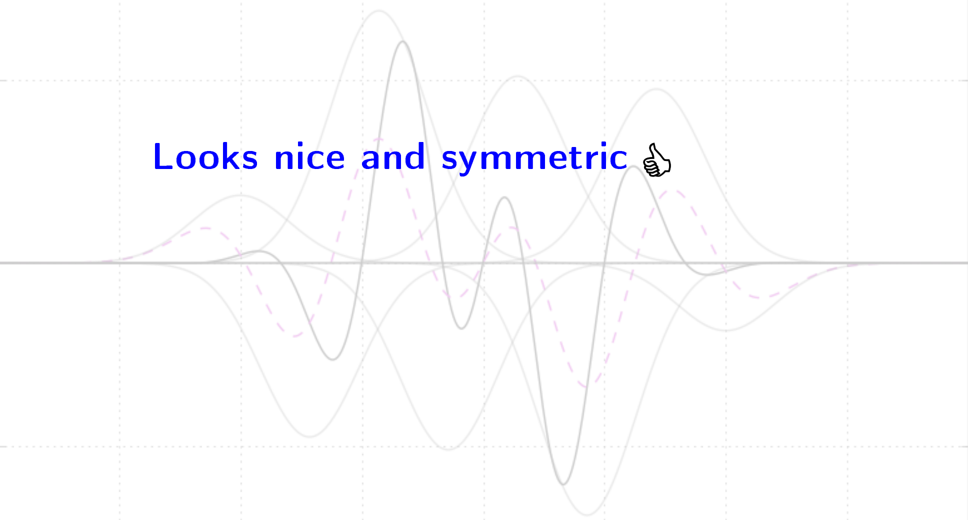


The RBF method



Looks nice and symmetric 👍

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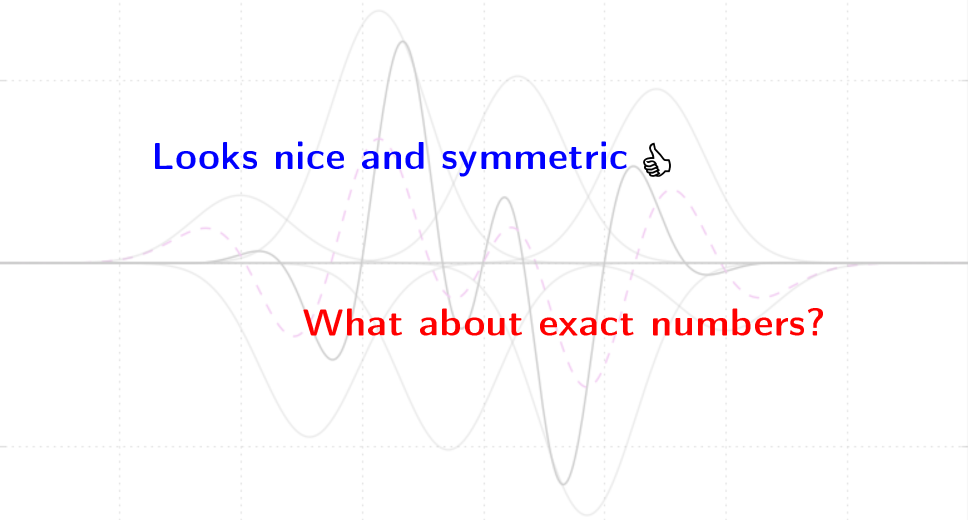

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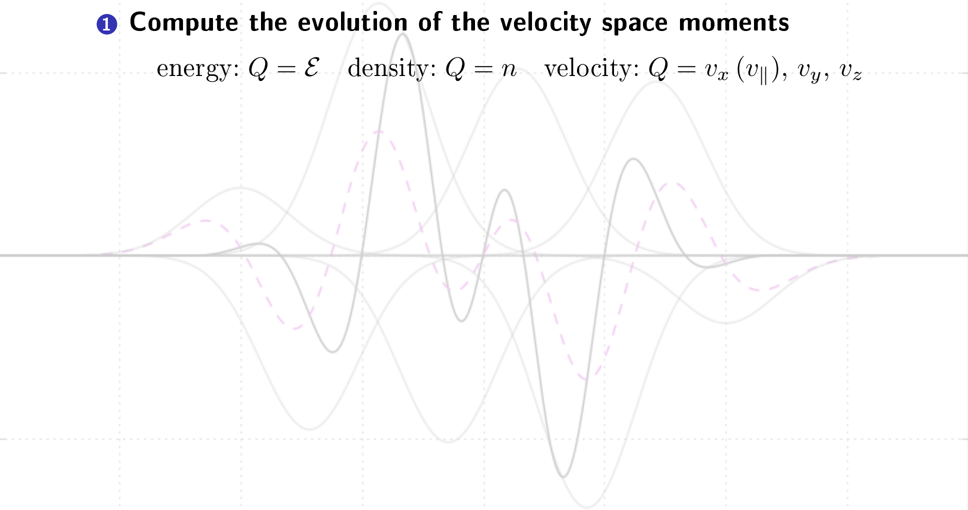
What about exact numbers?

Conservation properties

1. **Compute the evolution of the velocity space moments**

energy: $Q = \mathcal{E}$ density: $Q = n$ velocity: $Q = v_x\ (v_{\|}),\ v_y,\ v_z$



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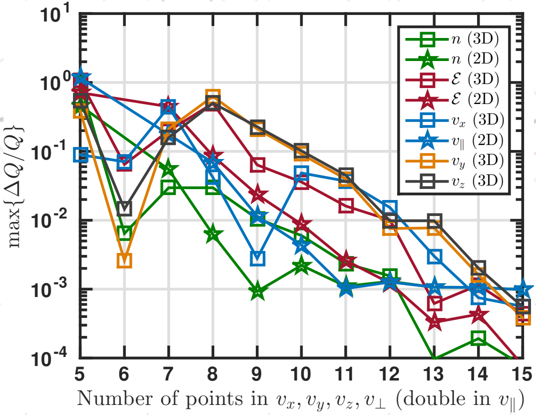

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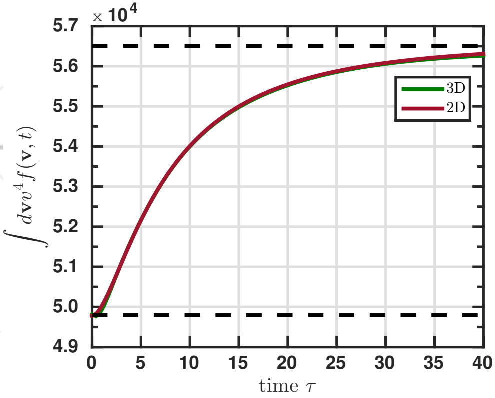

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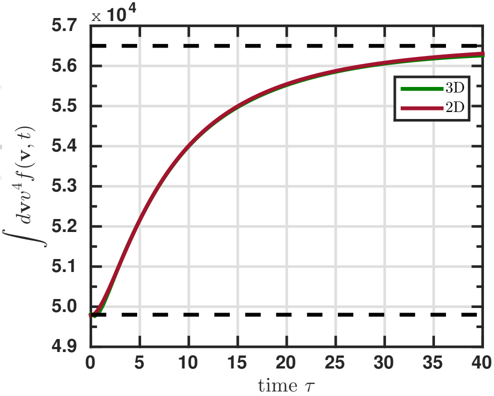

Not bad

Outline

MHD, gyrokinetics, you name it!

1. **Vlasov + collisions**

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \nabla f_s + \frac{e_s}{m_s} \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = \sum_{\bar{s}} C_{s\bar{s}}[f_s, f_{\bar{s}}] \ ,$$

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$$\nabla \cdot \boldsymbol{E} = (1/\varepsilon_0) \sum_s \int d\boldsymbol{v} \, f_s(\boldsymbol{v}, t)$$

$$\nabla \times \boldsymbol{B} = \mu_0 \sum_s e_s \int d\boldsymbol{v} \, \boldsymbol{v} f_s(\boldsymbol{v}, t) + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}$$

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RBF already tamed the collision operator.
What about the rest?

RBF discretization of the Vlasov-Maxwell system

1. **RBF Vlasov-Fokker-Planck**

$$\sum_{j} \mathcal{M}_s^{ij}\, \mathcal{L}_s^{ij} w_s^j = \sum_{k,\ell,\bar{s}} w_s^k\, w_{\bar{s}}^\ell\, C_{s\bar{s}}^{ik\ell}, \quad \forall \quad i \in 1, 2, 3, \ldots, \qquad (1)$$

where $\mathcal{M}_s^{ij}$ is the RBF matrix as previously and the operator $\mathcal{L}_s^{ij}$ is

$$\mathcal{L}_s^{ij} \doteq \frac{\partial}{\partial t} + \boldsymbol{v}_s^i \cdot \nabla + 2\gamma_s^j \frac{e_s}{m_s} \left[(\boldsymbol{v}_s^j - \boldsymbol{v}_s^i) \cdot \boldsymbol{E} + (\boldsymbol{v}_s^j \times \boldsymbol{v}_s^i) \cdot \boldsymbol{B} \right]$$

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Resemblance to moment equations...

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$$\text{Momentum flux tensor} \quad \mathbf{\Pi}_s = \sum_i w_s^i \, m_s \left[\mathcal{T}_s^i \, \mathbb{I} + \mathbf{v}_s^i \mathbf{v}_s^i \right]$$

$$\text{Energy flux} \quad \mathbf{Q}_s = \sum_i w_s^i \, \mathbf{v}_s^i \left[\frac{5}{2} \mathcal{T}_s^i + \frac{1}{2} m_s (v_s^i)^2 \right]$$

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$$

Easy to compute even for a large number of RBFs

Outline

1. What are Radial Basis Functions (RBFs)?
2. Discretization of the Collision operator
3. Non-linear relaxation problem
4. What about the Vlasov-Maxwell part?
5. **Summary and where to proceed**

Summary

Introduction to RBFs

- How to interpolate scattered data
- Convergence of the interpolation

New idea to address the collision operator

- Gaussian RBF expansion for the distributions
- Analytic expansion for the non-linear Fokker-Planck operator

Demonstration of a non-linear relaxation problem

- Both 2-D and 3-D RBF methods seem to work
- Conservation properties reasonable
- Consistent with analytical results

Where to proceed

Applied mathematics

- Well-known conditioning problem with the RBF matrix
- Experiment with different mesh configurations
- Galerkin projection instead of the center collocation

Physics

- Heating scenarios
- Loss cones
- Non-linear stuff

Vlasov-Maxwell-Fokker-Planck

- Multispecies scenarios
- Advection terms
- Coupling to field solvers